COURSE CERTIFIED BY:
COURSE CERTIFIED THRU:

SY6301
Navigation and State Estimation

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Terminal Learning Objectives

By the end of this course you should be able to:

- Apply foundational navigation concepts to air and space systems
- Construct and analyze navigation measurement and dynamic models (GPS, INS, AltNav)
- Implement and evaluate estimation methods (Least Squares, Kalman Filter, EKF)
- Assess navigation performance, integrity, and fault behavior using quantitative metrics
- Synthesize and communicate data-driven conclusions for navigation systems under test

Silv's ROEs

- These are my slides. Flag anything that could be sharper.
- We're going from zero to hero in 9 hours. Full understanding requires both attention and interaction. Stay engaged.
- Ask questions early and often. Slow me down anytime.
- I've been a TPS student myself, so I know you're constantly making priority calls. Side-task when you must, be considerate of others, and live with the consequences of what you choose to focus on.
- Slides are not meant to stand alone. Take notes actively.
- Use the course website to get deeper insights and reference materials.
- I'm on Slack 24/7 for questions or discussion.

Course Deliverables

Deliverable	Type	Weight	Due
Project PRR	Group	50%	Friday 12 June, 0800
Homework	Individual	50%	Sunday 14 June, 2359

SY6301 – Navigation and State Estimation

Block 1: Introduction, Frames, and Errors

Block Objectives

By the end of this block you should be able to:

- Explain what a navigation system estimates
- Distinguish between **truth**, **measurement**, and **estimate**
- Identify core frames used in navigation: **ECEF**, **NED**, **Body**
- Define position error and compute **horizontal** / **vertical** metrics
- Explain why frame consistency is required before statistics or integrity logic

What is Navigation?

- **Engineering definition:** Navigation estimates vehicle **state** relative to a chosen **reference frame**
- Typical state vector:

$$\hat{\mathbf{x}} = \begin{bmatrix} \text{position} \\ \text{velocity} \\ \text{attitude} \end{bmatrix}$$

- Inputs are imperfect: sensors, models, and assumptions contain noise and bias
- A navigation solution is therefore:

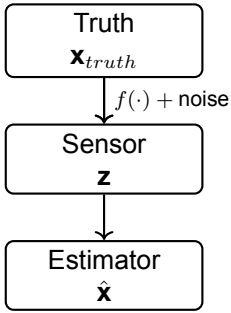
$$(\hat{\mathbf{x}},\ \mathbf{P})$$

  where **P** is the **state error covariance** (uncertainty)
- Together, they enable guidance, control, mission effectiveness, and safety

Truth vs Measurement vs Estimate

- **Truth**: $\mathbf{x}_{truth}$
- **Measurement**: $\mathbf{z} = f(\mathbf{x})$
- **Estimate**: $\hat{\mathbf{x}}$
- **Estimation error:**

$$\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}_{truth}$$



$$\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}_{truth}$$

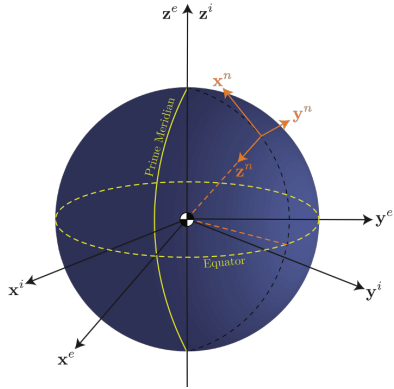
Why Frames Matter

- Suppose someone says: `error = [2, 1, -0.5]`
- You must ask:
  - In what frame? ECEF? NED? Body?
  - What are the units?
  - Where is the origin/reference defined?

Navigation computations only make sense with consistent frames.

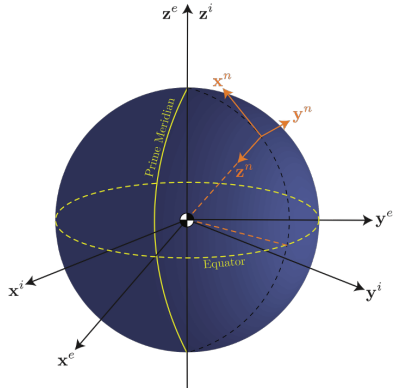
Frame 1: ECEF (Earth-Centered, Earth-Fixed)

- **Global reference:** Cartesian frame attached to the rotating Earth
- Axes:
  - X: toward lat $0°$, lon $0°$ (equator, Greenwich)
  - Y: toward lat $0°$, lon $90°$E (equator, east)
  - Z: toward North Pole (Earth spin axis)
- Origin at Earth's center of mass
- Natural for GNSS geometry and global trajectories
- Less intuitive for local error interpretation



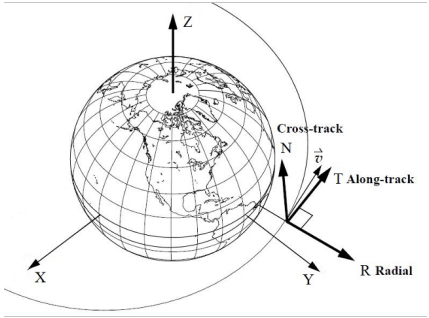
Frame 2: Local-Level NED (North-East-Down)

- **Local reference:** Tangent frame at a chosen reference LLH
- Axes:
  - N: north
  - E: east
  - D: down
- Assumes locally flat plane near the reference point
- Standard for aviation navigation and error metrics
- Great for plotting position error, velocity error, and covariances



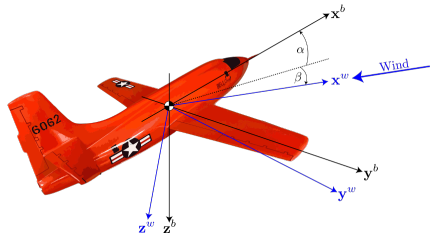
Frame 3: Radial-Transverse-Normal (RTN)

- **Local orbital reference:**
  Spacecraft-centered, oriented to position and velocity in inertial space
- Axes:
  - R: radial, outward from Earth's center
  - T: along-track, tangent to orbit in direction of motion
  - N: cross-track, normal to orbital plane (right-hand rule)
- Rotates with orbital motion; time-varying angular rate
- Standard for orbital relative motion (rendezvous, proximity ops)



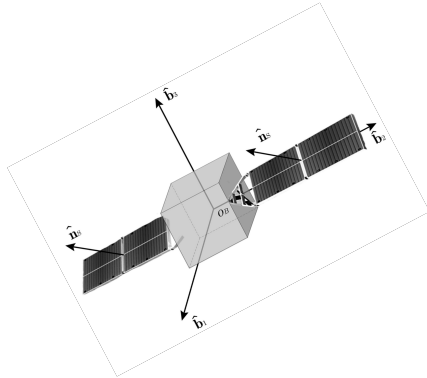
Frame 4: Aircraft Body Frame (Vehicle-Fixed, FRD)

- **Sensor reference:** Frame rigidly attached to the vehicle
- Axes:
  - X: forward
  - Y: right
  - Z: down
- IMU accelerometers measure specific force in body
- IMU gyroscopes measure body-to-NED attitude



Frame 5: Spacecraft Body Frame (sc-frame)

- **Vehicle-fixed:** Frame attached to the spacecraft structure; carries attitude, torque, and sensor pointing
- Axes (mission-dependent):
  - X: instrument boresight or nadir-pointing
  - Y: along solar array or side panel
  - Z: completes the right-hand rule
- Rotates and translates with the spacecraft
- Reference for star trackers, gyros, sun sensors, thrusters



Frame Transformations: Direction Cosine Matrices (DCMs)

- **Goal:** Express the *same physical vector* in a different frame
- **Convention:** DCM $\mathbf{C}_a^b$ transforms vectors from frame a to frame b

$$\mathbf{v}^b = \mathbf{C}_a^b \, \mathbf{v}^a$$

- **DCM properties:**
  - Orthonormal: $\mathbf{C}^\top \mathbf{C} = \mathbf{I}$
  - Inverse equals transpose:

$$\left(\mathbf{C}_a^b\right)^{-1} = \left(\mathbf{C}_a^b\right)^\top = \mathbf{C}_b^a$$

Rotation moves components, not the physical vector.

Frame Transformations: 2D DCM Example

- Consider two frames rotated by angle θ
- Components change, vector does not

$$\mathbf{v}^b = \mathbf{C}_a^b(\theta)\,\mathbf{v}^a$$

- 2D rotation matrix:

$$\mathbf{C}_a^b(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

- Key intuition:
  - Same arrow in space
  - Different numerical representation

Demo: 2D Direction Cosine Matrix

MATLAB version in `code/DcmExample.m`

Frame Transformations: 3D DCMs and Moving Points

- 3D generalization:

$$\mathbf{v}_b = \mathbf{C}_a^b \mathbf{v}_a, \qquad \mathbf{C}_a^b \in \mathbb{R}^{3\times 3}$$

- One parameterization: **Roll–Pitch–Yaw**

$$\mathbf{C}_a^b(\phi,\theta,\psi) = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\phi)$$

  - Order matters

- **Vectors vs Points:**
  - Vectors: rotation only
  - Points: rotation + translation

- Point transform:

$$\mathbf{p}_b = \mathbf{C}_a^b\,(\mathbf{p}_a - \mathbf{r}_{a\to b})$$

Error Definitions in NED

- Error vector can be defined in NED frame
- Intuitive for operational and test reporting
- Connects directly to targeting requirements

$$\mathbf{e}_{NED} = \begin{bmatrix} e_N \\ e_E \\ e_D \end{bmatrix}$$

$$e_H = \sqrt{e_N^2 + e_E^2}, \qquad e_V = |e_D|$$

Quick Exercise

$$\mathbf{p}_{truth} = \begin{bmatrix} 1000 \\ 500 \\ -50 \end{bmatrix}, \quad \hat{\mathbf{p}} = \begin{bmatrix} 1003 \\ 498 \\ -49 \end{bmatrix}$$

- Compute $\mathbf{e}_{NED}$
- Compute e_H and e_V
- If rotated into ECEF, what changes? What does not?

Demo: Target Location Error (TLE)

- **Mission goal:** Quantify targeting error from an airborne sensor
- Express both positions in the same frame:
  - Sensor-derived target LLH $\rightarrow$ ECEF $\rightarrow$ NED
  - True target LLH $\rightarrow$ ECEF $\rightarrow$ NED
- Compute the NED error vector: $\mathbf{e}_{NED} = \hat{\mathbf{p}}_{NED} - \mathbf{p}_{NED}$
- Reduce to operational metrics (e_H and e_V)
- **Interpretation:** Horizontal and vertical miss distance

Demo: Target Location Error

MATLAB version in `code/TargetLocationError.m`

Wrap-Up

- Navigation estimates vehicle **state** relative to a chosen **reference frame**, paired with a **covariance P** that quantifies uncertainty
- **Truth**, **measurement**, and **estimate** are distinct quantities; the estimation error is $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}_{truth}$
- Three core frames: **ECEF** (global Cartesian), **NED** (local-level tangent), **Body** (vehicle-fixed)
- **DCMs** transform vectors between frames; points additionally require a translation
- Position error reduces to operational metrics: $e_H = \sqrt{e_N^2 + e_E^2}$ and $e_V = |e_D|$
- Frame consistency is a **prerequisite** for any statistics or integrity logic that follows

Next block: where do those position errors come from? Inertial sensors integrate small errors into growing drift.

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Block 2: Inertial Error Growth and State Propagation

Block Objectives

By the end of this block you should be able to:

- Explain why inertial navigation errors **grow over time**
- Understand how small sensor errors propagate into **velocity and position**
- Identify common error models used in navigation systems
- Describe the concept of **state propagation**
- Interpret inertial drift using simple error metrics

What an IMU Measures

- Inertial Measurement Unit (IMU) provides:
  - **Specific force** from accelerometers
  - **Angular rate** from gyroscopes
- Measurements are expressed in the **body frame**
- Inertial navigation algorithms integrate these measurements to estimate:
  - Velocity
  - Position
  - Attitude

The Inertial Error Integration Chain

- Small sensor errors propagate through the “mechanization” equations
- Accelerometer error produces:
  - Velocity error (single integration)
  - Position error (double integration)
- Gyroscope error produces:
  - Attitude error
  - Incorrect projection of acceleration into NED frame
  - Further velocity and position error

Inertial drift is the result of integrating imperfect measurements.

Common Inertial Error Types

- **Bias**
  - Constant offset in a measurement
  - Produces linearly growing velocity error
- **Random noise**
  - Sample-to-sample variation
  - Produces random walk behavior
- **Drift**
  - Accumulated navigation error over time

Simple Error Models

- To estimate navigation performance, we use simplified models
- Example: Velocity bias random walk

$$b_{v,k+1} = b_{v,k} + w_k$$

- Example: Correlated noise (Gauss-Markov)

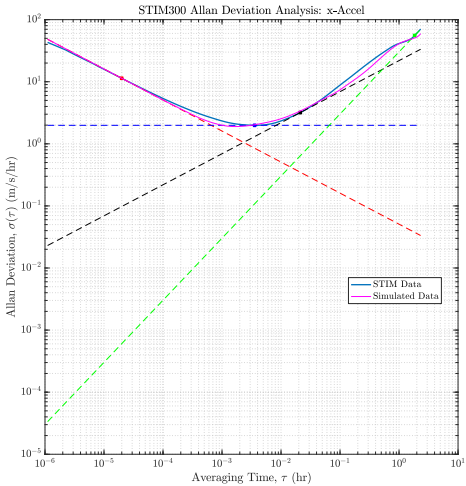
$$x_{k+1} = e^{-dt/T} x_k + w_k$$

- These models capture the **statistics** of sensor behavior

Error models capture statistics without modeling every physical detail.

Allan Variance: Characterizing IMU Errors on the Ground

- **Ground test, not flight test:** long static IMU run, typically ~ 6 hours
- Separates the noise processes in the IMU sensors
- **Procedure:** compute Allan deviation $\sigma(\tau)$ over cluster times τ and read slopes off the log-log plot
  - $-1/2$: white noise (angle/vel RW)
  - 0: bias instability
  - $+1/2$: rate random walk
- **Why we care:** numerical values feed straight into the Kalman filter as **Q** and as bias dynamics (random walk or Gauss-Markov)



Navigation State

- Navigation algorithms estimate a **state vector**

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \\ \mathbf{a} \end{bmatrix}$$

- Where
  - **p** = position (3×1)
  - **v** = velocity (3×1)
  - **a** = attitude between body and NED frames (3×1)
- The estimator also maintains an **uncertainty estimate**:

$$\mathbf{P} = \text{covariance of the state estimate } (9 \times 9)$$

State Propagation

- Navigation algorithms repeatedly **propagate** the state forward in time

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}_k$$

- Where
  - $\mathbf{x}_k$ = current state
  - $\mathbf{u}_k$ = IMU sensor measurements
  - $\mathbf{w}_k$ = process noise
- The estimator must also propagate the **covariance**

$$\mathbf{P}_{k+1} = \mathbf{F}_k \mathbf{P}_k \mathbf{F}_k^\top + \mathbf{Q}_k$$

- This describes how **uncertainty grows over time** as we propagate

Quick Exercise

Assume a constant accelerometer bias produces a constant velocity error.

- What happens to position error over time?
- Does the error grow linearly or quadratically?
- What happens if the system receives periodic corrections?

Why Inertial Drift Tests Are Long

- Inertial navigation errors evolve over time
- Horizontal inertial errors often exhibit a natural oscillation called the **Schuler cycle**

$$T_{Schuler} \approx 84.4 \text{ minutes}$$

- To properly characterize inertial drift, test runs should cover at least one full cycle
- Practical rule of thumb in flight test:

Inertial drift runs should be at least 85 minutes

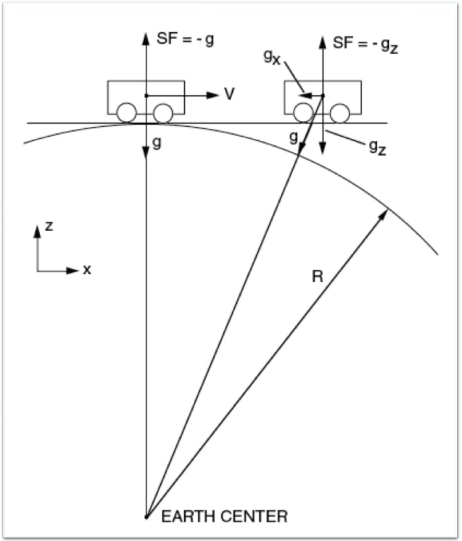
Good inertial testing requires observing a full Schuler cycle.

Why the Schuler Cycle Happens

- Consider a small **attitude error** in the navigation solution
- Gravity is now projected slightly into the horizontal plane

$$a_H \approx g\,\theta$$

- This produces a horizontal acceleration error
- The navigation solution begins to drift horizontally
- The system eventually **overshoots and corrects**, producing an oscillation



Demo: Inertial Drift

- Simulated inertial navigation with no external aiding
- Observe horizontal error growth
- Compute drift slope over time
- Interpret drift in operational units (NM/hr)

Demo: Inertial Drift

MATLAB version in `code/InertialDriftDemo.m`

Wrap-Up

- Inertial navigation integrates sensor measurements to estimate motion
- Small sensor errors accumulate through integration
- Drift is an unavoidable property of inertial systems without external updates
- Navigation algorithms propagate a state forward in time using motion models
- Schuler cycle drives required length of flight tests for inertial-only requirements

Next block: if inertial drift is unavoidable, can a second sensor bound it? $\rightarrow$ **optimal fusion**.

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Block 3: Optimal Fusion

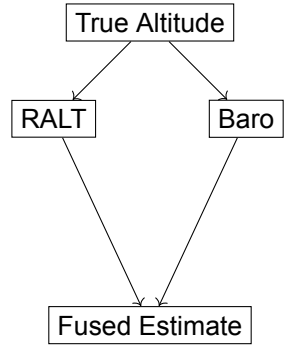
Block Objectives

By the end of this block you should be able to:

- Explain why combining sensors improves navigation estimates
- Compute a fused estimate using **weighted averaging**
- Understand why optimal weights depend on **measurement uncertainty**
- Compute the **variance of a fused estimate**
- Recognize that the **Kalman filter is recursive optimal fusion**

Why Fuse Sensors?

- Navigation systems often measure the same quantity using **multiple sensors**
- Example: altitude measurement
  - Radar altimeter: 100 ± 2 m
  - Barometric altitude: 100 ± 10 m
- Question: what altitude estimate should we use?
  - Choose one sensor?
  - Average them?
  - Something smarter?



Simple Averaging

Suppose two sensors measure the same quantity

$$z_1, \qquad z_2$$

A simple approach is the average

$$\hat{x} = \frac{z_1 + z_2}{2}$$

- Works if both sensors have **similar uncertainty**
- But what if

$$\sigma_1 = 1, \qquad \sigma_2 = 10$$

- Should both sensors influence the estimate equally?

Weighted Averaging

Instead of a simple average we can assign **weights**

$$\hat{x} = w_1 z_1 + w_2 z_2$$

subject to

$$w_1 + w_2 = 1$$

- Weights determine how much each sensor influences the estimate
- Sensors with lower uncertainty **should** receive a larger weight

Optimal Fusion: Problem Setup

Start with a weighted estimate

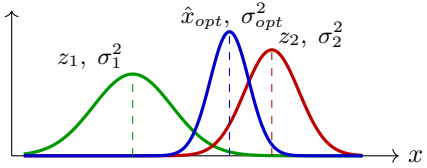
$$\hat{x} = w_1 z_1 + w_2 z_2$$

Assume measurement variances

$$Var(z_1) = \sigma_1^2, \qquad Var(z_2) = \sigma_2^2$$

The variance of the fused estimate is

$$Var(\hat{x}) = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2$$



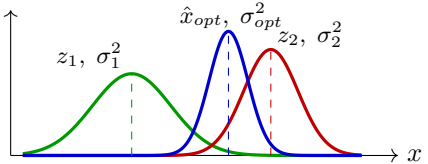
Optimal Fusion: Problem Setup

Goal: choose w_1, w_2 to minimize

$$Var(\hat{x}) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2$$

subject to

$$w_1 + w_2 = 1$$



Solving the minimum-variance problem yields

$$w_1 = \frac{1/\sigma_1^2}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}, \qquad w_2 = \frac{1/\sigma_2^2}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$$

Optimal Fusion Result

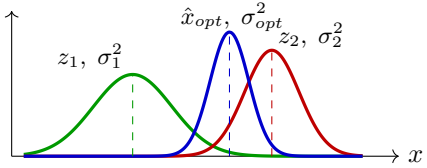
The optimal fused estimate becomes

$$\hat{x}_{opt} = \frac{\frac{z_1}{\sigma_1^2} + \frac{z_2}{\sigma_2^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$$

The variance of the fused estimate is

$$\sigma_{opt}^2 = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{-1}$$

- Sensors with lower variance receive higher weight
- The fused estimate has lower uncertainty than either sensor alone



Optimal fusion reduces estimation uncertainty.

Information Interpretation

Define **information**

$$I = \frac{1}{\sigma^2}$$

Then the fused estimate becomes

$$I_{total} = I_1 + I_2$$

Interpretation:

- Each sensor contributes **information**
- Total information is the **sum of individual information**

Quick Exercise (2 minutes)

Two sensors measure the same quantity.

$$z_1 = 100, \qquad \sigma_1 = 2$$

$$z_2 = 110, \qquad \sigma_2 = 6$$

Use optimal fusion

$$\hat{x}_{opt} = \frac{\frac{z_1}{\sigma_1^2} + \frac{z_2}{\sigma_2^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$$

- Compute the fused estimate $\hat{x}$
- Which sensor influences the estimate more?

Demo: Sensor Fusion

Simulate two noisy sensors

- Sensor 1: small noise
- Sensor 2: large noise

Fuse them using inverse-variance weighting

$$\hat{x} = \frac{\frac{z_1}{\sigma_1^2} + \frac{z_2}{\sigma_2^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$$

Observe how the fused estimate reduces uncertainty.

Demo: Optimal Fusion

MATLAB version in `code/OptimalFusionDemo.m`

From Fusion to Kalman Filtering

Sensor fusion above assumes all measurements arrive **at once.**

Navigation systems instead process measurements **sequentially:**

$$\text{predict} \rightarrow \text{measure} \rightarrow \text{update}$$

This leads to the recursive update

$$\hat{x}_{new} = \hat{x}_{prior} + K(z - \hat{x}_{prior})$$

where

$$K = \frac{P}{P + R} \qquad \text{“Optimal Weight / Kalman Gain”}$$

- P = prior ($\hat{x}_{prior}$) uncertainty
- R = measurement (z) uncertainty

Wrap-Up

- Multiple sensors can be combined to improve navigation estimates
- Optimal fusion weights measurements using **inverse variance**
- Fusion reduces estimate uncertainty
- Recursive optimal fusion leads to the **Kalman filter**

Next block: fuse repeatedly as new measurements arrive → **recursive (Kalman) fusion**.

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Block 4: Scalar Kalman Filter

Block Objectives

By the end of this block you should be able to:

- Explain how the Kalman filter performs **recursive optimal fusion**
- Understand the roles of **prediction**, **measurement**, and **innovation**
- Interpret the **Kalman gain** as an uncertainty-weighted trust factor
- Understand how **state and covariance** are updated each step
- Prepare for multi-state navigation filters in the next lesson

Recursive Estimation Problem

Navigation systems operate **continuously over time**.

At each time step we have:

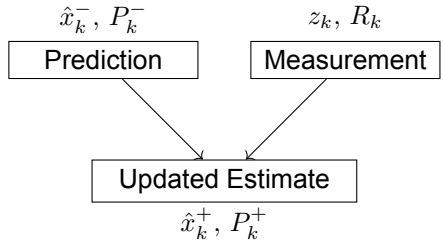
- A **prediction** and it's **uncertainty**:

$$\hat{x}_k^-, \quad P_k^-$$

- A **measurement** and it's **uncertainty**:

$$z_k, \quad R_k$$

Goal: Fuse prediction and measurement **optimally** to produce: $\hat{x}_k^+, \quad P_k^+$



Kalman Filter Update

The Kalman filter updates the estimate using

$$\hat{x}_k^+ = \hat{x}_k^- + K_k \underbrace{\left(z_k - \hat{x}_k^-\right)}_{\text{innovation}}$$

Where

- $\hat{x}_k^-$ = predicted state
- z_k = measurement
- $z_k - \hat{x}_k^-$ = innovation
- K_k = Kalman gain

Innovation

The **innovation** measures the difference between prediction and measurement.

$$\nu_k = z_k - \hat{x}_k^-$$

Interpretation:

- Small innovation $\rightarrow$ prediction was good
- Large innovation $\rightarrow$ measurement strongly corrects the estimate

Innovation measures the surprise in the measurement.

Kalman Gain

The scalar Kalman gain is

$$K_k = \frac{P_k^-}{P_k^- + R_k}$$

Where

- P_k^- = prediction uncertainty
- R_k = measurement variance

Interpretation:

- Large $R \rightarrow$ measurement noisy $\rightarrow$ small gain
- Large $P^- \rightarrow$ prediction uncertain $\rightarrow$ large gain

Kalman gain determines how much we trust the measurement.

Covariance Update

After incorporating the measurement

$$P_k^+ = (1 - K_k)P_k^-$$

Meaning:

- Uncertainty **decreases** after measurement update
- The filter estimates both the **state** and its **uncertainty**

Prediction Step

Before receiving the next measurement, we propagate the estimate forward.
State prediction:

$$\hat{x}_{k+1}^{-} = F\hat{x}_{k}^{+}$$

Covariance prediction:

$$P_{k+1}^{-} = FP_{k}^{+}F^{T} + Q_{k}$$

Where

- F = state transition model
- Q = process noise covariance

Update from previous time (k) becomes the prior for next time $(k+1)$

Scalar Kalman Filter Loop

Prediction: $k \rightarrow k+1$

$$\hat{x}_{k+1}^{-} = F\hat{x}_{k}^{+}$$

$$P_{k+1}^{-} = FP_{k}^{+}F^{T} + Q$$

Update: $- \rightarrow +$

$$K_{k+1} = \frac{P_{k+1}^{-}}{P_{k+1}^{-} + R}$$

$$\hat{x}_{k+1}^{+} = \hat{x}_{k+1}^{-} + K_{k+1}\left(z_{k+1} - \hat{x}_{k+1}^{-}\right)$$

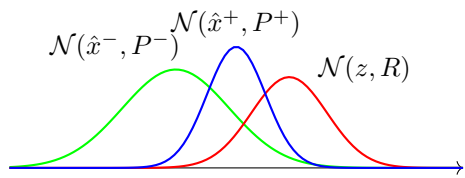
$$P_{k+1}^{+} = (1 - K_{k+1})P_{k+1}^{-}$$

Gaussian Interpretation

Prediction and measurement each provide information.

Kalman filtering fuses them to produce a posterior estimate.

- $\mathcal{N}(\hat{x}^-, P^-) \rightarrow$ Prior distribution
- $\mathcal{N}(z, R) \rightarrow$ Measurement distribution
- $\mathcal{N}(\hat{x}^+, P^+) \rightarrow$ Posterior distribution



Quick Exercise

Prediction

$$\hat{x}^- = 1000 \text{ m}$$

$$P^- = 25 \text{ m}^2$$

Measurement

$$z = 1010 \text{ m}$$

$$R = 9 \text{ m}^2$$

Compute

- Kalman gain
- Updated estimate (x^+) and variance (P^+)

Demo: Scalar Kalman Filter

Scenario: A train moves along a track at approximately constant velocity.

Simulation

- True train position evolves over time
- A sensor provides **noisy position measurements**
- A scalar Kalman filter estimates the train position

Key observations

- The filter smooths noisy measurements
- The estimate converges toward the true trajectory
- The covariance decreases as confidence grows

Demo: Scalar Kalman Filter

MATLAB version in `code/KalmanFilterScalar.m`

Wrap-Up

Key ideas

- Kalman filtering performs **recursive optimal fusion**
- The **innovation** measures the difference between prediction and measurement
- The **Kalman gain** determines how much the measurement influences the estimate
- The filter updates both the **state** and its **uncertainty**

Next block

- Multi-state Kalman filters for navigation systems
- Position, velocity, and attitude estimation (9-dimensional)

SY6301 – Navigation and State Estimation

Block 5: Multi-State Kalman Filter for Navigation

Block Objectives

By the end of this block you should be able to:

- Why navigation problems require **multi-state estimation**
- How the scalar Kalman filter generalizes to **vector states**
- The role of the **state transition matrix F**
- The role of the **measurement matrix H**
- How navigation filters simultaneously estimate **2D position and velocity**

Navigation filters estimate multiple states simultaneously

Why Multiple States?

The scalar Kalman filter estimated a **single variable**.

Navigation systems must estimate many variables simultaneously:

- Position
- Velocity
- Attitude
- Sensor biases
- Clock errors

The Kalman filter propagates and updates a state vector

State Vector Representation

Instead of estimating a single scalar, we estimate a **vector of states**. For a 2D tracking problem:

$$\mathbf{x}_k = \begin{bmatrix} p_{x,k} \\ p_{y,k} \\ v_{x,k} \\ v_{y,k} \end{bmatrix}$$

Where

- $p_{x,k},\ p_{y,k}$ = position in x and y at time step k
- $v_{x,k},\ v_{y,k}$ = velocity in x and y at time step k

State Dynamics Model

Assume constant-velocity motion in both axes.

$$p_{x,k+1} = p_{x,k} + v_{x,k}\Delta t$$

$$p_{y,k+1} = p_{y,k} + v_{y,k}\Delta t$$

$$v_{x,k+1} = v_{x,k}, \quad v_{y,k+1} = v_{y,k}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In matrix form:

$$\mathbf{x}_{k+1} = \mathbf{F}\mathbf{x}_k$$

Process Noise

- Real systems deviate from constant-velocity motion
- Unknown accelerations inject uncertainty into p_x, p_y, v_x, v_y
- Modeled as independent noise σ_a^2 on each axis

$$\mathbf{Q} = \sigma_a^2 \begin{bmatrix} \frac{\Delta t^4}{4} & 0 & \frac{\Delta t^3}{2} & 0 \\ 0 & \frac{\Delta t^4}{4} & 0 & \frac{\Delta t^3}{2} \\ \frac{\Delta t^3}{2} & 0 & \Delta t^2 & 0 \\ 0 & \frac{\Delta t^3}{2} & 0 & \Delta t^2 \end{bmatrix}$$

Process noise models uncertainty in the motion model

Measurement Model

Suppose the sensor (e.g., GPS) measures **2D position only**.

The measurement equation is:

$$\mathbf{z}_k = \begin{bmatrix} p_{x,k} \\ p_{y,k} \end{bmatrix} + \boldsymbol{\nu}_k$$

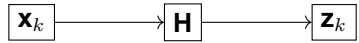
In matrix form:

$$\mathbf{z}_k = \mathbf{H}\mathbf{x}_k + \boldsymbol{\nu}_k$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

R is now a 2×2 matrix:

$$\mathbf{R} = \begin{bmatrix} \sigma^2_{p_x} & 0 \\ 0 & \sigma^2_{p_y} \end{bmatrix}$$



Multi-State Kalman Filter Equations

Propagate:

$$\hat{\mathbf{x}}_{k+1}^{-} = \mathbf{F}\hat{\mathbf{x}}_{k}^{+}$$

$$\mathbf{P}_{k+1}^{-} = \mathbf{F}\mathbf{P}_{k}^{+}\mathbf{F}^{T} + \mathbf{Q}$$

Update:

$$\mathbf{K}_{k} = \mathbf{P}_{k}^{-}\mathbf{H}^{T}\left(\mathbf{H}\mathbf{P}_{k}^{-}\mathbf{H}^{T} + \mathbf{R}\right)^{-1}$$

$$\hat{\mathbf{x}}_{k}^{+} = \hat{\mathbf{x}}_{k}^{-} + \mathbf{K}_{k}\left(\mathbf{z}_{k} - \mathbf{H}\hat{\mathbf{x}}_{k}^{-}\right)$$

$$\mathbf{P}_{k}^{+} = \left(\mathbf{I} - \mathbf{K}_{k}\mathbf{H}\right)\mathbf{P}_{k}^{-}$$

Note: $\mathbf{R}$ is now a 2×2 matrix and $\mathbf{K}_{k}$ is 4×2.

Covariance Matrix Interpretation

The covariance matrix **P** describes **uncertainty in each state** and correlations between states.

$$\mathbf{P} = \begin{bmatrix} \sigma_{p_x}^2 & \sigma_{p_x p_y} & \sigma_{p_x v_x} & \sigma_{p_x v_y} \\ \sigma_{p_y p_x} & \sigma_{p_y}^2 & \sigma_{p_y v_x} & \sigma_{p_y v_y} \\ \sigma_{v_x p_x} & \sigma_{v_x p_y} & \sigma_{v_x}^2 & \sigma_{v_x v_y} \\ \sigma_{v_y p_x} & \sigma_{v_y p_y} & \sigma_{v_y v_x} & \sigma_{v_y}^2 \end{bmatrix}$$

Interpretation:

- Diagonal terms = uncertainty in each state (p_x, p_y, v_x, v_y)
- Off-diagonal terms = correlation between state errors (e.g., x-position and x-velocity)
- x and y cross-terms are zero for independent axes

Navigation Example

Operational navigation filters estimate many states simultaneously.

Example: **INS/GPS 15-state navigation filter**

$$\mathbf{x}_k = \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} x & y & z & v_x & v_y & v_z & \phi & \theta & \psi & b_{ax} & b_{ay} & b_{az} & b_{gx} & b_{gy} & b_{gz} \end{array}\right]^T$$

Where:

- $(x,\ y,\ z)$ = position
- $(v_x,\ v_y,\ v_z)$ = velocity
- $(\phi,\ \theta,\ \psi)$ = attitude
- $\mathbf{b}_a$ = accelerometer biases
- $\mathbf{b}_g$ = gyroscope biases

Demo: 4-State Navigation KF

We will simulate:

- A vehicle moving in 2D with approximately constant velocity
- Noisy GPS position measurements in x and y
- A 4-state Kalman filter

The filter simultaneously estimates:

- 2D position: p_x, p_y
- 2D velocity: v_x, v_y

Demo: 4-State Kalman Filter

MATLAB version in `code/KalmanFilter4D.m`

Wrap-Up

- States form the vector $\hat{\mathbf{x}} = [p_x,\ p_y,\ v_x,\ v_y]^T$
- The 4×4 motion model is defined by **F**
- Measurements are mapped by the 2×4 matrix **H**
- Uncertainty is captured by the 4×4 covariance matrix **P**
- Measurement noise is now the 2×2 matrix **R**

Next block: measurements have been linear so far. GPS pseudoranges are not. $\rightarrow$ **GPS fundamentals**.

Navigation filters estimate PVA, sensor errors, and more!

SY6301 – Navigation and State Estimation

Block 6: GPS Fundamentals for Navigation

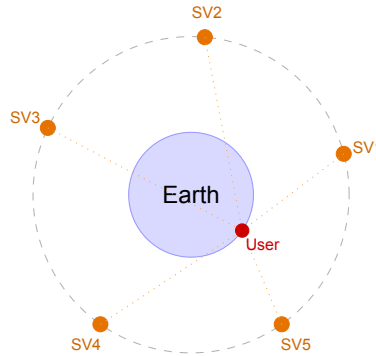
Block Objectives

By the end of this block you should be able to:

- Describe the GPS constellation and signal architecture
- Explain how a **pseudorange** is formed and why it is *pseudo*
- Formulate GPS positioning as a **nonlinear least-squares problem**
- Explain how satellite geometry amplifies pseudorange noise into position error
- Identify the four dominant GPS error sources and their typical magnitudes
- Interpret GPS accuracy specifications (CEP, 1σ, 2DRMS)

The GPS Constellation

- 24+ satellites in medium Earth orbit
  - Altitude ≈20,200 km Period ≈12 hours
- ≥ 4 satellites in view from anywhere on Earth at any time
- GPS is one of several constellations; collectively called **GNSS** (also GLONASS, Galileo, BeiDou)
- Each satellite broadcasts its precise position and an atomic timestamp



How GPS Measures Distance: The Pseudorange

- Satellite transmits a PRN code at known time t_{tx}; receiver records arrival at t_{rx}
- Apparent range: $c \cdot (t_{\text{rx}} - t_{\text{tx}})$
- **Problem:** receiver clock not synchronized with satellite $\rightarrow$ unknown bias δt_u corrupts every measurement

The result is a **pseudorange**:

$$\tilde{\rho}_i = \underbrace{\|\mathbf{r}_u - \mathbf{r}^{(i)}\|}_{\text{true range}} + c\,\delta t_u + \epsilon_i$$

This is a **nonlinear function** of user position $\mathbf{r}_u$

Pseudorange = true range + clock bias + noise

GPS as a Nonlinear Estimation Problem

- Four unknowns: $[x,\ y,\ z,\ \delta t_u]$
- Each satellite contributes one equation:

$$\tilde{\rho}_i = h_i(\mathbf{x}) + \epsilon_i, \qquad h_i(\mathbf{x}) = \|\mathbf{r}_u - \mathbf{r}^{(i)}\| + c\,\delta t_u$$

- Need ≥ 4 satellites for a unique solution
- Cannot be solved in closed form — solved iteratively (Newton-Raphson, Levenberg-Marquardt, MATLAB `fitnlm`)

GPS positioning is a nonlinear least-squares problem solved iteratively

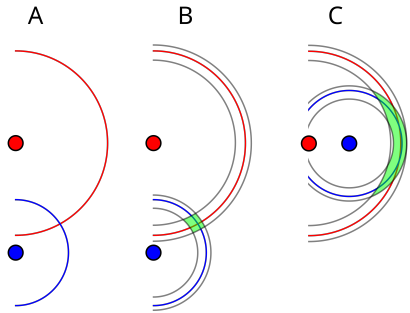
How Satellite Geometry Affects Accuracy

Each pseudorange defines a **sphere** centered on a satellite.

Position = intersection of spheres.

- **Well-spread:** spheres cross at a sharp angle → tight intersection → small uncertainty
- **Clustered:** spheres nearly parallel → poorly constrained → large uncertainty

Same σ_ρ — very different position accuracy



Dilution of Precision (DOP)

DOP quantifies the geometry effect:

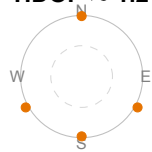
$$\sigma_{\text{position}} = \text{DOP} \times \sigma_\rho$$

Variants: **HDOP**, **VDOP**, **PDOP**, **TDOP**

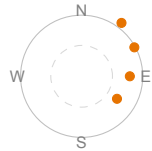
- HDOP ≈ 1–2: good geometry
- HDOP > 4: degraded accuracy

DOP is a **property of geometry only**

HDOP ≈ 1.2



HDOP ≈ 4.5



Geometry matters as much as pseudorange noise

GPS Error Sources

Four dominant contributors to pseudorange noise σ_ρ:

Source	Typical Magnitude	Mitigation
Ionospheric delay	1–5 m	Dual-frequency; model correction
Tropospheric delay	0.5–2 m	Saastamoinen model
Multipath	0.5–3 m	Antenna placement; signal processing
Receiver noise	0.1–0.3 m	Hardware quality

Lumped result: $\sigma_\rho \approx 3$–5 m (L1 SPS) $\longrightarrow$ sets diagonal of **R** in the Kalman filter

GPS accuracy is bounded by physics, not just hardware

GPS Limitations

- **Jamming:** signal at Earth's surface is very weak (≈ -130 dBm) — easily overpowered by a small transmitter
- **Spoofing:** counterfeit signals mislead the receiver; a growing and accessible threat
- **Denied environments:** indoors, underground, urban canyons, heavy foliage
- **Vertical accuracy:** always worse than horizontal (VDOP $>$ HDOP due to geometry)

GPS works until it doesn't — integration with INS fills the gaps

Quick Exercise

A GPS receiver tracks satellites with $\sigma_\rho = 3$ m.

Scenario A: Satellites well spread $\longrightarrow$ HDOP = 1.5

Scenario B: Satellites clustered $\longrightarrow$ HDOP = 4.0

- Compute horizontal 1σ position accuracy for each scenario
- A solar storm doubles σ_ρ to 6 m. What happens in each scenario?

$$\sigma_{\text{pos}} = \text{HDOP} \times \sigma_\rho$$

Demo: GPS Positioning

Scenario: 2D GPS positioning from pseudorange measurements

Simulation

- Simulate pseudoranges with realistic noise ($\sigma_\rho = 3$ m)
- Solve for position using nonlinear least-squares (Gauss-Newton)

Key observations

- Run with good geometry vs. poor geometry
- Visualize position uncertainty ellipses from `fitnlm` covariance
- Compare to DOP $\times\, \sigma_\rho$ prediction

Demo: GPS Positioning

MATLAB version in `code/GPSPositioningDemo.m`

Wrap-Up

- GPS measures **pseudoranges** corrupted by clock bias — not direct position
- Positioning requires ≥ 4 satellites to solve for $[x,\ y,\ z,\ \delta t_u]$ — a **nonlinear** problem
- Satellite geometry (**DOP**) amplifies pseudorange error: $\sigma_{\text{pos}} = \text{DOP} \times \sigma_\rho$
- Error sources (ionosphere, troposphere, multipath) set σ_ρ and the **R** matrix
- GPS has operational limits — inertial integration fills the gaps

Next block: $h_i(\mathbf{x})$ is nonlinear — how do we feed GPS into a KF built for linear measurements? $\rightarrow$ **Extended Kalman Filter**

Next block: extending the KF to handle nonlinear sensors

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Block 7: Extended Kalman Filter for GPS

Block Objectives

By the end of this block you should be able to:

- Distinguish between **loose coupling** and **tight coupling** GPS/INS integration
- Apply a standard Kalman filter update when GPS provides a **linear position measurement**
- Explain why pseudorange measurements require an **Extended Kalman Filter**
- Compute the **pseudorange Jacobian H** from geometry
- Trace through one complete **EKF predict-and-update cycle** with GPS pseudoranges

Two Ways to Fuse GPS into a Filter

Loose Coupling

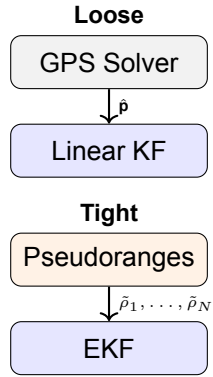
GPS solver runs first; filter receives solved position $\hat{\mathbf{p}}$

- Measurement: $\mathbf{z} = \hat{\mathbf{p}} + \boldsymbol{\nu}$
- Observation matrix: $\mathbf{H} = [\mathbf{I} \mid \mathbf{0}]$ picks off positions
- Standard linear KF update — no extra math

Tight Coupling

Filter ingests raw pseudoranges directly

- Measurement per satellite: $\tilde{\rho}_i = \|\mathbf{p} - \mathbf{s}^{(i)}\| + b$
- Nonlinear in $\mathbf{x}$ — requires linearization
- Extended Kalman Filter (EKF)



Loose Coupling: GPS as a Position Sensor

GPS solver outputs solved position $\hat{\mathbf{p}}$ — the filter treats it as a noisy measurement

$$\mathbf{z}_k = \hat{\mathbf{p}}_k + \boldsymbol{\nu}_k, \qquad \boldsymbol{\nu}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$$

For our **2D example** (state $= [p_x,\ p_y,\ v_x,\ v_y]^T$ from Block 5):

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \text{(same } \mathbf{H} \text{ as Block 5)}$$

In **3D** the state grows to $[p_x, p_y, p_z, v_x, v_y, v_z]^T$ and $\mathbf{H} = [\mathbf{I}_3 \mid \mathbf{0}_{3\times3}]$ — same idea, bigger matrix

R is set from GPS accuracy spec — $\sigma^2_{GPS} \approx (\text{HDOP} \times \sigma_\rho)^2$

Tight Coupling: The Challenge

Each satellite contributes one pseudorange measurement

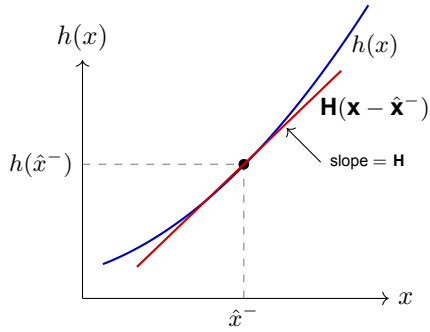
$$\tilde{\rho}_i = \underbrace{\|\mathbf{p} - \mathbf{s}^{(i)}\|}_{\text{nonlinear in } \mathbf{p}} + b + \epsilon_i$$

- The standard KF update **requires** a linear model $\mathbf{z} = \mathbf{H}\mathbf{x} + \boldsymbol{\nu}$
- There is no matrix **H** that exactly represents a range measurement
- **Solution:** linearize $h_i(\mathbf{x})$ around the current predicted state $\hat{\mathbf{x}}^-$

EKF Key Idea: Linearization

Replace the nonlinear function with its **first-order Taylor expansion**:

$$h(\mathbf{x}) \approx h(\hat{\mathbf{x}}^-) + \underbrace{\left.\frac{\partial h}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}^-}}_{\mathbf{H}} (\mathbf{x} - \hat{\mathbf{x}}^-)$$



The **innovation** uses the nonlinear function:

$$r = \tilde{\rho} - h(\hat{\mathbf{x}}^-)$$

Then the standard KF equations apply with $\mathbf{H}$ evaluated at $\hat{\mathbf{x}}^-$

Re-linearize at every time step — this is the EKF loop

Extending the State for GPS

To use pseudoranges directly we add the **receiver clock bias** b as a state. For our **2D example** (one new state on top of Block 5):

$$\mathbf{x} = \begin{bmatrix} p_x & p_y & v_x & v_y & b \end{bmatrix}^T \quad \text{(5 states)}$$

- Clock bias b [m] is the range-equivalent receiver timing error
- Modeled as slowly varying (random walk or Gauss-Markov)
- Every pseudorange equation includes b — the EKF estimates it automatically

Real-world 3D: add p_z, $v_z \Rightarrow$ 7-state vector $[p_x, p_y, p_z, v_x, v_y, v_z, b]^T$ — same structure

Adding one state unlocks full tight-coupling GPS integration

The Pseudorange Jacobian

Measurement function for satellite i:

$$h_i(\mathbf{x}) = \|\mathbf{p} - \mathbf{s}^{(i)}\| + b = \rho_i + b$$

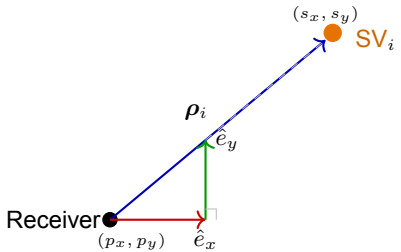
Partial derivatives give the row $\mathbf{H}_i = \dfrac{\partial h_i}{\partial \mathbf{x}}$.

2D example (5-state $[p_x, p_y, v_x, v_y, b]^T$):

$$\mathbf{H}_i = \Big[\underbrace{\hat{e}_x,\ \hat{e}_y}_{\text{unit vec.}},\ 0,\ 0,\ 1\Big]_{1\times 5}$$

where $\hat{e}_x = \dfrac{p_x - s_x^{(i)}}{\rho_i}, \quad \hat{e}_y = \dfrac{p_y - s_y^{(i)}}{\rho_i}$

3D extends to $[\hat{e}_x, \hat{e}_y, \hat{e}_z, 0, 0, 0, 1]_{1\times 7}$



$\mathbf{H}_i$ points from receiver to satellite – geometry matters!

EKF Predict-and-Update Loop

Propagate (same as linear KF):

$$\hat{\mathbf{x}}^- = \mathbf{F}\hat{\mathbf{x}}^+ \qquad \mathbf{P}^- = \mathbf{F}\mathbf{P}^+\mathbf{F}^T + \mathbf{Q}$$

Update — one satellite at a time:

$$r_i = \tilde{\rho}_i - h_i(\hat{\mathbf{x}}^-) \qquad \text{(nonlinear innovation)}$$

$$\mathbf{K}_i = \mathbf{P}^-\mathbf{H}_i^T\left(\mathbf{H}_i\mathbf{P}^-\mathbf{H}_i^T + \sigma_\rho^2\right)^{-1}$$

$$\hat{\mathbf{x}}^+ = \hat{\mathbf{x}}^- + \mathbf{K}_i\, r_i \qquad \mathbf{P}^+ = \mathbf{P}^- - \mathbf{K}_i\mathbf{H}_i\mathbf{P}^-$$

Repeat the update step for each satellite in view

Quick Exercise: Full EKF Update Step

Given (5-state: $[p_x,\ p_y,\ v_x,\ v_y,\ b]^T$):

$$\hat{\mathbf{x}}^- = \begin{bmatrix} 0 \\ 0 \\ 10 \\ -5 \\ 20 \end{bmatrix} \text{m}, \quad \mathbf{P}^- = 100\,\mathbf{I}_5, \quad \mathbf{s} = \begin{bmatrix} 3000 \\ 4000 \end{bmatrix} \text{m}, \quad \tilde{\rho} = 5035 \text{ m}, \quad \sigma_\rho^2 = 100 \text{ m}^2$$

- **Linearize:** Compute $\rho = \|\hat{\mathbf{p}}^- - \mathbf{s}\|$, predicted pseudorange $h = \rho + b$, and Jacobian $\mathbf{H} = [\hat{e}_x,\ \hat{e}_y,\ 0,\ 0,\ 1]$
- **Innovate:** Compute $r = \tilde{\rho} - h(\hat{\mathbf{x}}^-)$
- **Gain:** Compute $\mathbf{K} = \mathbf{P}^-\mathbf{H}^T\left(\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \sigma_\rho^2\right)^{-1}$
- **Update:** Compute $\hat{\mathbf{x}}^+ = \hat{\mathbf{x}}^- + \mathbf{K}\,r$. Which states changed? Why?

Quick Exercise: Full EKF Update Step (Workspace)

Given: $\hat{\mathbf{x}}^- = [0,\ 0,\ 10,\ -5,\ 20]^T$ m, $\mathbf{P}^- = \text{diag}(100)$, $\mathbf{s} = [3000,\ 4000]^T$ m, $\tilde{\rho} = 5035$ m, $\sigma_\rho^2 = 100$ m^2

Demo: EKF with GPS Pseudoranges

Scenario: Vehicle navigates in 2D; GPS pseudoranges update an EKF

State model (5 states)

- 2D position, 2D velocity
- Receiver clock bias

Key observations

- EKF relinearizes **H** at every step as state estimate changes
- One sequential update per satellite

Demo: Extended Kalman Filter

MATLAB version in

`code/ExtendedKalmanFilter5D.m`

Wrap-Up

- **Loose coupling:** GPS solver outputs position; linear $\mathbf{H} = [\mathbf{I} \mid \mathbf{0}]$ — standard KF applies
- **Tight coupling:** pseudoranges fed directly; nonlinear $h_i(\mathbf{x})$ requires linearization $\rightarrow$ EKF
- The **pseudorange Jacobian** $\mathbf{H}_i = [\hat{e}_x,\ \hat{e}_y,\ \ldots,\ 1]$ encodes satellite geometry
- EKF **relinearizes** at each step: innovation uses $h(\hat{\mathbf{x}}^-)$, Kalman gain uses $\mathbf{H}|_{\hat{\mathbf{x}}^-}$
- Adding clock bias b to the state closes the loop — no separate GPS solver needed

Next block: the EKF assumes well-behaved sensors. What happens when one lies? $\rightarrow$ **fault detection and integrity**.

The EKF is optimal fusion applied to nonlinear sensors

SY6301 – Navigation and State Estimation

Block 8: Fault Detection, Integrity, and HMI

Block Objectives

By the end of this block you should be able to:

- Define a **sensor fault** and explain why it causes the Kalman filter innovation to grow
- Apply the **Mahalanobis distance test** to determine when an innovation is statistically suspicious
- Distinguish between **filter error bounds** (from **P**) and **integrity bounds** (protection levels)
- Define **Hazardous Misleading Information (HMI)** and identify it from position error and protection level time histories

What Is a Sensor Fault?

A fault is any condition where the measurement no longer behaves according to the assumed model:

$$\mathbf{z}_k = h(\mathbf{x}_k) + \mathbf{v}_k, \qquad \mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$$

Common fault sources in GPS/navigation:

- **Spoofing / jamming** — intentional range bias injected by an adversary
- **Unmodeled bias** — satellite clock anomaly, thermal drift, multipath
- **Wrong model parameters** — incorrect lever arm, timing offset, scale factor
- **Geometry collapse** — satellite dropout, poor PDOP

Key point: a fault is not just “bad noise” — it *violates the distributional assumptions* of the estimator, so the filter ingests it silently and corrupts its state estimate.

The Innovation Under a Fault

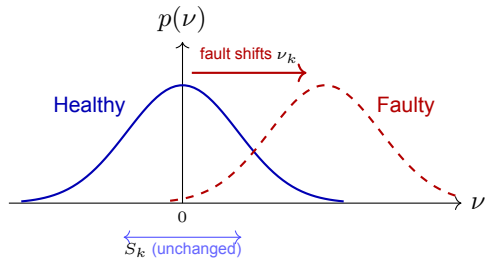
Under **healthy** conditions, the innovation is zero-mean:

$$\boldsymbol{\nu}_k = \mathbf{z}_k - h(\hat{\mathbf{x}}_k^-) \sim \mathcal{N}(\mathbf{0}, \mathbf{S}_k)$$

Under a **ramp fault** on one pseudorange, $z_k = h(x_k) + v_k + b_f(t_k)$, so the innovation becomes:

$$\nu_k \approx \underbrace{v_k}_{\text{noise}} + \underbrace{b_f(t_k)}_{\substack{\text{growing} \\ \text{bias}}}$$

- $S_k = \mathbf{H}\mathbf{P}^-\mathbf{H}^T + \sigma_\rho^2$ **does not change**
- ν_k grows steadily while S_k stays small



Result: Filter is confidently wrong.

How Big Is Too Big? — Mahalanobis Distance

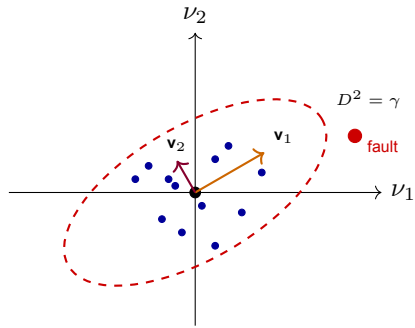
In **1D**, checking if a residual is suspicious is simple:

$$\frac{|\nu|}{\sigma} = \text{number of “sigmas”}$$

With **vector measurements** we need a generalization that accounts for covariance **S**.

Mahalanobis distance:

$$D^2 = \boldsymbol{\nu}^T \mathbf{S}^{-1} \boldsymbol{\nu}$$



D^2 measures how statistically surprising the innovation is, regardless of dimensions.

- Weights each residual component by its uncertainty
- Accounts for correlations between components

Detection as a Hypothesis Test

Under healthy conditions, D^2 follows a chi-squared distribution:

$$D^2 = \boldsymbol{\nu}^T \mathbf{S}^{-1} \boldsymbol{\nu} \;\sim\; \chi^2(m)$$

where m is the number of measurements stacked in $\boldsymbol{\nu}$.

Decision rule:

$$D^2 \underset{\text{healthy}}{\overset{\text{fault}}{\gtrless}} \gamma, \qquad \gamma = \chi^2_{m,\,1-P_{FA}}$$

Threshold γ is chosen from the desired **false alarm probability** P_{FA}:

Measurements m		$P_{FA} = 1\%$	$P_{FA} = 0.3\%$
1	(single pseudorange)	6.63	8.83
4	(batch, 4 satellites)	13.28	16.25

Quick Exercise: Is This Residual Suspicious?

Scenario: Satellite 1 has a ramp fault with rate $\dot{b}_f = 3$ m/s injected at $t_0 = 10$ s. The innovation variance is $S = 5^2 = 25$ m^2 (from $S = \mathbf{H}\mathbf{P}^{-}\mathbf{H}^T + \sigma_\rho^2$).

- **Detection:** At $t = 15$ s the pseudorange innovation is $\nu = 15$ m. Compute $D^2 = \nu^2/S$. Is this suspicious at $P_{FA} = 0.3\%$ ($m = 1$, $\gamma = 8.83$)?
- **Time-to-detect:** The bias grows linearly from zero. At what time does D^2 first exceed $\gamma = 8.83$? What is the time-to-detect $T_D = t_D - t_0$?
- **Trade-off:** If you tighten the threshold to $\gamma = 4.0$ (more sensitive), what happens to T_D? What is the cost?

Quick Exercise: Is This Residual Suspicious? (Workspace)

Given: $\dot{b}_f = 3$ m/s, $t_0 = 10$ s, $\sigma_S = 5$ m, $\gamma = 8.83$ ($m = 1$, $P_{FA} = 0.3\%$)

What Happens After Detection?

Detection alone is not enough — the system must **respond**.

Common fault responses:

- **Fault exclusion** — remove the offending satellite; continue with remaining SVs
- **Fault accommodation** — inflate **R** for the suspect SV, or augment state with a bias term
- **Fault recovery** — reintroduce the sensor once its innovations return to nominal
- **Multi-filter / parallel hypothesis** — run N sub-filters each excluding one SV; the sub-filter that remains statistically consistent identifies the culprit

The F-47 ANS uses a multi-filter architecture: one main filter plus sub-filters each excluding one sensor. This produces a **position integrity output** that is separate from the filter covariance **P**.

Navigation Integrity vs. Accuracy

These answer fundamentally different questions:

Accuracy	How close is my estimate to the truth?
Integrity	Can I *trust* this estimate right now?

Operationally, integrity is a system's ability to:

- Detect when its performance is unacceptable
- Alert the operator within a specified time
- Bound the risk of an **undetected failure**

Integrity is not about average performance;
it is a guarantee about worst-case behavior.

Error Bounds vs. Integrity Bounds — Part 1

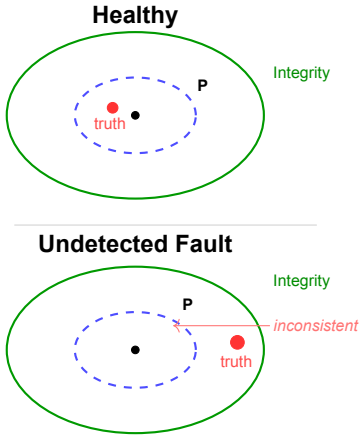
Filter Error Bound (from P)

- Derived from filter covariance: $\mathbf{P}_{pos} \Rightarrow$ error ellipse
- Interpretation: *"If the model is correct, true error lies here with high probability"*
- An undetected fault corrupts $\hat{\mathbf{x}}$ but does *not* grow **P**

Inconsistency:

$$\text{Actual error} \not\sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

The filter stays "confidently wrong" — the **P** ellipse no longer contains truth.



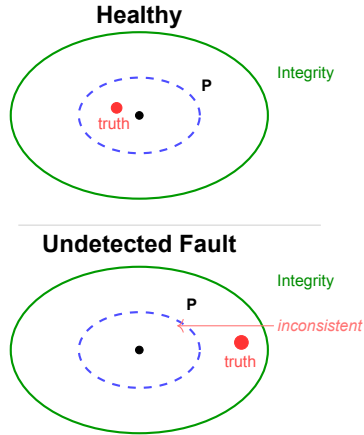
Error Bounds vs. Integrity Bounds — Part 2

Integrity Bound

- A separate, more conservative bound from the multi-filter architecture
- **Remains valid even when one sensor is faulty**
- Expressed as spatial thresholds: HPL and VPL

Healthy system: truth inside both **P** and integrity bound

Faulty system: truth outside **P** (inconsistent), but *still inside* integrity bound — no HMI yet



Protection Levels

The integrity bound is expressed as usable spatial thresholds — the **protection levels**.

Horizontal Protection Level (HPL):

$$\text{HPL} = n_\alpha \sqrt{P_{\text{int},N} + P_{\text{int},E}}$$

Vertical Protection Level (VPL):

$$\text{VPL} = n_\alpha \sqrt{P_{\text{int},D}}$$

- P_{int} = diagonal elements of the **integrity covariance** (not the filter covariance **P**)
- n_α = multiplier for desired containment probability (e.g., $n_\alpha = 2.576$ for 99%)
- HPL/VPL are **not** confidence ellipses from the EKF — they are guaranteed performance zones from the multi-filter architecture, valid under a single sensor fault

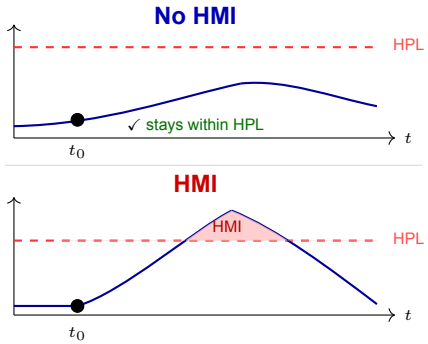
Hazardous Misleading Information (HMI)

Definition: HMI occurs when the **true position error exceeds the protection level** while the fault is undetected:

$\text{HMI}_H: \ e_H > \text{HPL} \qquad \text{HMI}_V: \ |e_D| > \text{VPL}$

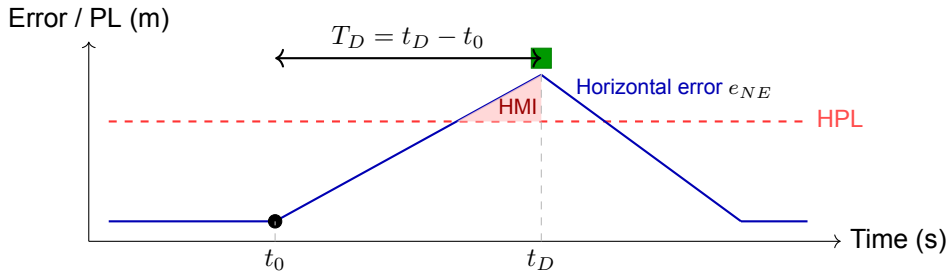
- **Safe outcome:** error grows but stays within HPL (integrity bound covers the fault scenario)
- **HMI outcome:** error exceeds HPL before or during the detection window — system actively misleads the operator

HMI is the most critical failure mode in nav systems.



Reading the Plots: Measuring TTD and HMI Exposure

As tester, you observe position error and protection level over time:



HMI exposure = total time in $[t_0, t_D]$ where $e_{NE} >$ HPL (or $|e_D| >$ VPL)

Demo: EKF with an Injected Fault

Using EKF demo from last time:

- Ramp bias injected on satellite 1 at `t_fault`
- Compute D_k^2 per satellite update; compare to threshold
- Plot sat innovations — one grows, others stay small
- Plot position error vs. HPL; shade HMI exposure window

Things to observe:

- How does D^2 change before vs. after the fault starts?
- How does pos error compare to the reported **P** ellipse?

Demo: Fault Detection

MATLAB version in `code/FaultDetectionDemo.m`

Wrap-Up

- A **sensor fault** violates the measurement model — the filter ingests it silently and corrupts its state estimate
- The **innovation** $\boldsymbol{\nu}_k$ carries a growing bias under a fault; the **Mahalanobis distance** $D^2 = \boldsymbol{\nu}^T \mathbf{S}^{-1} \boldsymbol{\nu}$ measures how statistically surprising it is
- Fault detection is a **hypothesis test:** $D^2 \gtrless \gamma$, where γ trades false alarm rate against detection sensitivity
- **Integrity** $\neq$ **accuracy:** the filter covariance **P** becomes misleading under a fault; the **integrity bound** (HPL/VPL) remains valid
- **HMI** occurs when the position error exceeds the protection level — the most critical navigation failure mode

Next block: apply everything to a real test plan $\rightarrow$ **F-47 ANS Project**.

SY6301 – Navigation and State Estimation

Block 9: Project Introduction

Block Objectives

By the end of this block you should be able to:

- Explain why correlated time-series data requires **effective sample size** N_{eff}, not raw sample count N
- Estimate T_{corr} from an autocorrelation function and explain the **10% rule of thumb**
- Derive N_{eff} from first principles and **plan a statistically defensible test duration**
- Apply the **95th-percentile ECDF** for accuracy requirements
- Estimate inertial **drift rate** with a confidence interval on the slope
- Infer **fault detection time** from navigation error behavior without FDE flags

F-47 ANS — Background and Mission Need

The F-47 Navigation Computer has been upgraded with an **Alternative Navigation Suite (ANS)** to sustain performance in **GPS-denied environments**.

The ANS fuses inertial measurements with:

- GPS (when available)
- AltNav aiding: MagNav, VisNav, SoPNav

When GPS is denied, the ANS transitions to AltNav-aided operation while continuing to provide **integrity-relevant uncertainty outputs**.

F-47 ANS — Program Status

HITL testing is complete. Key findings:

- Horizontal navigation error has $T_{\text{corr}} \approx 15$ s during steady-state operation
  $\Rightarrow$ statistical foundation for planning effective independent sample size
- Over 200 fault events executed across step bias, ramp bias, and others
- **Most stressing case:** single-SV ramp bias—longest T_D, greatest HMI exposure
- DT will focus on SV ramp bias to bound worst-case fault detection performance

Critical DT constraint:

- SPO directed production-representative sensors and avionics bus data only
- No dedicated Flight Test Instrumentation–no internal fault flags
  $\Rightarrow$ detection time t_D must be *inferred* from navigation solution behavior

System Under Test — ANS Modes and Outputs

ANS Operating Modes

Mode	Sensor Suite
AllSource	Inertial + GPS + AltNav
AltNav	Inertial + AltNav (GPS denied)
Inertial	Inertial only

Truth position (LLH) collected via an independent truth reference system.

Outputs Available

- True position (LLH)
- Estimated position (LLH)
- Estimated position covariance (N-E-D)
- Integrity protection levels: HPL, VPL

These outputs form the basis of **all four compliance assessments**.

System Under Test — Datasets Provided

File	Mode	Purpose
`data_AllSource`	AllSource	Accuracy assessment
`data_AltNav`	AltNav	Accuracy assessment
`data_Inertial`	Inertial	Drift assessment
`data_Spoof`	AllSource	Fault detection / integrity

Each dataset contains: time (s), true position (LLH), estimated position (LLH), covariance (N-E-D), integrity (HPL/VPL), and fault onset times t_0 (Spoof only).

Critical warning: `data_Spoof` must **not** be used for accuracy assessment. Fault injection events produce episodically elevated errors that inflate the 95th-percentile estimate. **Accuracy and fault-detection testing require separate dedicated runs.**

F-47 ANS Performance Requirements

Req	Mode	Threshold	Sample Requirement
1	AllSource	$H_{95} \leq 1.0$ m, $V_{95} \leq 5.0$ m	$N_{\text{eff}} \geq 300$
2	AltNav	$H_{95} \leq 5.0$ m, $V_{95} \leq 10.0$ m	$N_{\text{eff}} \geq 300$
3	Inertial	Drift rate ≤ 1.0 NM/hr (upper 95% CI)	$T \geq 85$ min
4	AllSource	$T_D \leq 5$ s, $\text{HMI}_H \leq 1$ s, $\text{HMI}_V \leq 1$ s	27 of 30 events

Reqs 1 & 2 require $N_{\text{eff}} \geq 300$ effective independent samples — the correlation structure of navigation error drives the minimum run duration.

Reqs 3 & 4 specify the statistical method directly: upper CI bound and 27/30 rule.

The Problem with Correlated Data

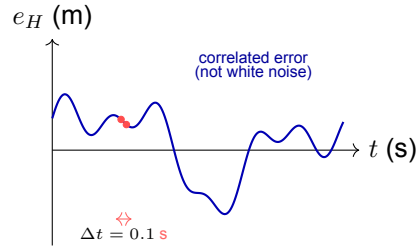
A 15-minute AllSource run at 10 Hz yields $N = 9{,}000$ samples.

Tempting conclusion:

$$\text{CI width} \propto \frac{1}{\sqrt{9{,}000}} \approx 0.011 \quad \text{(very tight!)}$$

The problem:

- Consecutive error samples are *not* independent
- $e_{k=1}$ and $e_{k=2}$ are essentially the same number
- Using $N = 9{,}000$ falsely assumes independence



Actual effective sample count is much smaller than $N = 9{,}000$.

The Autocorrelation Function

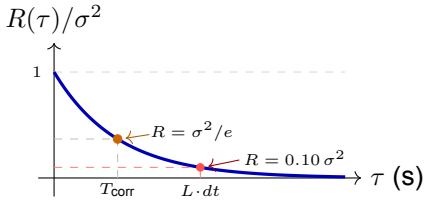
The **ACF** measures how similar a signal is to a time-shifted copy:

$$R(\tau) = \mathbb{E}\big[e(t)\, e(t+\tau)\big]$$

For navigation error driven by an AR(1) process:

$$\boxed{R(\tau) = \sigma^2\, e^{-|\tau|/T_{\text{corr}}}}$$

- T_{corr}: Correlation time — the "memory" of the error
- At $\tau = 0$: $R(0) = \sigma^2$ (variance)
- At $\tau = T_{\text{corr}}$: $R = \sigma^2/e \approx 0.37\,\sigma^2$



HITL result (given): $T_{\text{corr}} \approx 15\,\text{s}$

The 10% Rule of Thumb — How Close Is It?

Standard formula (from variance-of-the-mean derivation):

$$N_{\text{eff}} = \frac{T_{\text{total}}}{2\,T_{\text{corr}}}$$

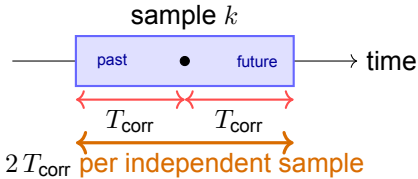
Method	Implicit ρ threshold	Difference
Standard (sum of ACF)	$\rho = e^{-2} \approx 13.5\%$	—
10% decimation rule	$\rho = 10.0\%$	-13%

The 10% rule gives an N_{eff} about 13% smaller relative to the standard formula.

Effective Sample Size — Where Does the Factor of 2 Come From?

The factor of 2 appears because the ACF sum is prefixed by 2 in the variance formula.

Physical interpretation: each sample is correlated with neighbors both **forward and backward** in time. The two-sided correlation window spans $2\,T_{\text{corr}}$ of the timeline.



Each “independent look” therefore occupies $2\,T_{\text{corr}}$ seconds:

$$N_{\text{eff}} = \frac{T_{\text{total}}}{2\,T_{\text{corr}}}$$

Test Duration Planning

Given: $T_{\text{corr}} = 15\,\text{s}$, $N_{\text{eff}} \geq 300$ required for accuracy assessments.

Solve for minimum run duration:

$$N_{\text{eff}} = \frac{T_{\text{total}}}{2\,T_{\text{corr}}} \geq 300 \quad \Longrightarrow \quad T_{\text{total}} \geq 2 \times 15 \times 300 = 9{,}000\,\text{s} \approx 150\,\text{min}$$

Run duration	N (at 10 Hz)	N_{eff}	Meets $N_{\text{eff}} \geq 300$?
15 min	9,000	30	**No**
60 min	36,000	120	**No**
150 min	90,000	300	**Yes**

95% Statistical Methods — Which Tool Matches the Requirement?

The F-47 ANS requirement reads:

"Horizontal position error shall be ≤ 1.0 *m (95% confidence)."*

- **95th-percentile ECDF**
  Find the value below which 95% of error samples fall:

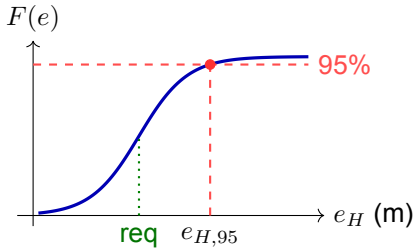
$$H_{95} = \texttt{prctile(errH, 95)}$$

- **95% CI on the mean**
  Tests whether the *mean* is below a bound. The requirement constrains the *tail*, not the mean.

Reading the ECDF

Steps to assess compliance:

- Compute horizontal errors e_H vs. truth
- Build the empirical CDF (`ecdf` in MATLAB)
- Read off $e_{H,95}$ — value where $F(e) = 0.95$
- Compare: **PASS** if $e_{H,95} \leq 1.0$ m



Inertial Drift — The Error Model

In inertial-only mode, velocity bias random walk **integrates** into position. Horizontal error grows approximately linearly:

$$e_H(t) = a + b\,t + \varepsilon(t)$$

Goal: estimate slope b (drift rate in m/s) and verify ≤ 1.0 NM/hr.

OLS estimate using MATLAB backslash:

$$\hat{\boldsymbol{\beta}} = \mathbf{X} \backslash \mathbf{e}_H, \qquad \mathbf{X} = \begin{bmatrix} \mathbf{1} & \mathbf{t} \end{bmatrix}$$

gives $\hat{\boldsymbol{\beta}} = [\hat{a},\ \hat{b}]^\top$ — intercept and slope.

Unit conversion:

$$\hat{b}\,[\text{m/s}] \times \frac{3600}{1852} = \hat{b}\,[\text{NM/hr}]$$

Inertial Drift — 95% CI and Pass Criterion

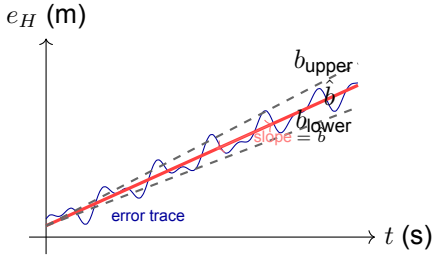
95% CI on slope $\hat{b}$:

$$\hat{b} \pm t_{N-2,\,0.975} \cdot \hat{\sigma}_b$$

Pass criterion:

$$b_{\text{upper}}\,[\text{NM/hr}] \leq 1.0$$

Compare the **upper bound**, not the point estimate. Even if $\hat{b}$ is below 1.0 NM/hr, a large uncertainty could still result in FAIL.



Fault Detection Without FTI — The Algorithm

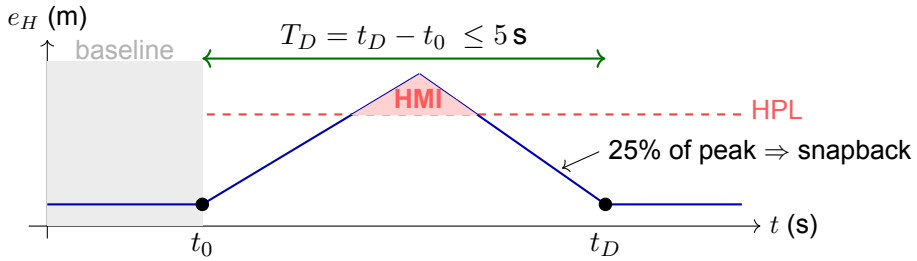
Constraint: production sensors only — no FTI, no internal fault flags. Fault onset time t_0 is known from the test log. Detection time t_D must be *inferred* from navigation error behavior.

Peak-and-snapback algorithm:

- Compute **pre-fault baseline** from error in window $[t_0 - T_{\text{pre}},\ t_0)$
- Find **peak error** e_{peak} in the post-fault search window
- Set **snapback threshold**: $e_{\text{thr}} = e_{\text{base}} + \alpha\,(e_{\text{peak}} - e_{\text{base}}), \quad \alpha = 0.25$
- t_D = first time after peak where $e_H \leq e_{\text{thr}}$ for ≥ 0.5 s

Once t_D is identified, compute HMI exposure during $[t_0, t_D]$ and check $T_D = t_D - t_0$.

Fault Detection Without FTI — Timeline and HMI



27/30 Compliance Rule

Requirement 4 states: for 30 independent spoof events, at least 27 of 30 shall satisfy:

Metric	Threshold
Time-to-detect $T_D = t_D - t_0$	$\leq 5\,\text{s}$
Horizontal HMI exposure during $[t_0,\, t_D]$	$\leq 1\,\text{s}$
Vertical HMI exposure during $[t_0,\, t_D]$	$\leq 1\,\text{s}$

The 27/30 threshold was established by the SPO to reflect a probabilistic performance specification. The system is not required to be perfect on every event, but must demonstrate reliable, consistent performance across the test campaign.

MATLAB Script Map — Five Focused Parts

Each script is self-contained. Work through them in order.

Script	Task	Req
`Project_DataRequest.m`	Compute N_{eff}, justify run durations	Planning
`Project_CorrelationTime.m`	ACF from data, estimate T_{corr}, verify N_{eff}	Planning
`Project_AccuracyECDF.m`	95th-pct ECDF, compare to bounds	1 & 2
`Project_InertialDrift.m`	OLS slope, 95% CI, NM/hr comparison	3
`Project_FaultDetection.m`	Infer t_D, HMI exposure, 27/30 count	4

Wrap-Up

- Correlated time series have far fewer **independent samples** than the raw count N suggests — accuracy assessments require $N_{\text{eff}} = T_{\text{total}}/(2\,T_{\text{corr}})$
- T_{corr} is read from the ACF; the **10% rule** provides a quick estimate within $\sim 13\%$ of the standard formula
- Accuracy is a **tail** property — assess with the **95th-percentile ECDF**, not a CI on the mean
- Inertial drift is the OLS **slope** of $e_H(t)$; compliance compares the **upper 95% CI bound** to the requirement
- Without FTI, t_D is inferred from the **peak-and-snapback** behavior of the navigation error
- The **27/30 rule** converts a probabilistic specification into a finite-sample pass/fail criterion

Statistical planning is the difference between data and evidence.

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